SIXTH CONGRESS OF A.P.S.M.S.

October 18 - 20, 1996

Venue : Department of Mathematics, Nagarjuna University.

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PROFESSOR KESIRAJU SATYANARAYAN ENDOWMENT LECTURE

ON

GEOMETRY

The First Endowment Lecture Delivered by :

Prof. S.K.LAKSHMANA RAO

"Saraswathy"

501, Garutman Park

Bangalore -560 004.

Endowment Instituted by :

Prof. N.CH.Pattabhi Ramacharyulu

in memory of his late son

Dr. N.CH.Raghavacharyulu.

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THE BEAUTY OF TRIANGLE GEOMETRY

Prof. S.K. Lakshmana Rao[1]

501, Saraswathy

Garutman Park

Bangalore 560 004.

Geometry is an attractive piece in the drawing room of the palace of the queen of sciences. It has been a beauty for ages and has an importance in the history of knowledge.

Classical Geometry as understood till about 1840, rests on the postulates of Euclid and Euclidean Geometry held its supreme position with no rivalry or challenge for almost twenty centuries from the time of Euclid. Since the middle of the nineteenth century the realm of Geometry started widening its base, what with the developments in Algebraic Geometry, Symplectic Geometry, Continuous Geometry, Ordered Geometry of Artin, Finite Geometry, Differential Geometry, Tensor

[1]Former Faculty :

Indian Institute of Science

Bangalore 560 012. &

Regional Engineering College

Warangal 506 004.

Geometry et al. One may however feel perplexed as to what is Triangle Geometry.

According to Felix Kline, geometry of the triangle (in the domain of Euclideam Geometry) in an extensive but closed field that has surfaced over a vast time scale in the work of many secondary teachers who were exclusively devoted to the many remarkable points, lines, circles and other curves that are associated with a triangle. The numerous results involving the centre of gravity, the attitudes, the angle bisectors, the in-circle, the ex-circles, the circum-circle fall into an orderly arrangement in a systematic development on considering the projective invariant theory of the five points represented by homogeneous coordinates in the form $(a_i, b_i, c_i)(i = 1, 2, 3)$ and the two imaginary circular points at infinity viz., $(1, \pm i, o)$ where $i = \sqrt{-1}$. One is wonderstruck that so simple and common-place a plane figure as the triangle has such inexhaustibility in its properties.

It is not easy to define the subject of Triangle Geometry logically but one can easily recognize that given an arbitrary triangle ABC, it is possible to find points, lines and curves that have remarkable properties vis a vis the triangle. e.g. circum-centre, ortho-centre, in-centre, centroid, Fermat points, Torricelli point, Gergonne point, Brocard points, Lemoine point, Euler line, nine points circle, et al.

In deed, more than a hundred such distinguished objects (points or lines or curves) have been identified. {c.f. C.H.Kimberling : central points and central lines

in the plane of a triangle - to appear in Mathematical Magazine}.

In their seminal article : Neuere Dreiecks Geometrie [Encyklopaedie der Mathematischen Wissensenaften, Vol. III, AB10 (1914), pp 1173-1276] G.Berkhan and W.FR.Meyer proposed the definition of Triangle Geometry as the study of distinguished points, lines circles and conics associated with a triangle. In his Enlanger Programm, Felix Klein declared that triangle geometry is the invariant theory of five points under the projective group. After extensive contribution by many mathematicians over the centuries, triangle geometry has emerged as a specifically individual topic of study in plane geometry since 1870. It is the study of 'distinguished objects' associated with an arbitrary triangle. In his 1900 book "La Recente Geometria del Triangolo", C.Alasia mentions 566 metric formulas relating to the triangle and its distinguished objects.

A well known theorem of Triangle Geometry involves the trisectors of the interior angles of an arbitiary triangle. It was shown by Morley (1899) that the three pairs of trisectors of the internal angles of an arbitrary triangle that are adjacent to the three sides, intersect at points which enclose an equilateral triangle. Thus an arbitrary triangle is always a Morley triangle. The result is an unexpected as it is beautiful. Trisection of an arbitrary angle by the use of a ruler and compass is not possible. However, highly accurate order computer graphics are now available and it is worth a deep study to see if Morley's theorem and the use of computer graphics

can make any impact in mesh cutting for the adoption of triangular elements in the numerical study of plane problems using the finite element techniques.

The nine-points circle theorem is another beautiful result of triangle geometry and it is known since 1820. It was Poincelet who first observed that the middle points of the three sides, the feet of the altitudes and the middle points of the three segments of the three altitudes between the orthocenter and the three vertices are concyclic. This circle is tangent to the incircle and each of the three ex-circle. There are at least 43 distinguished points on the nine-points circle.

NAPOLEON's THEOREM

Let ABC be an arbitrary triangle and let BXC, CYA, AZB be equilateral triangles erected on the sides BC, CA, AB respectively, such that each of X, Y, Z lies outside the triangle ABC. If L, M, N are the three centres of equilateral triangles, the triangle L, M, N is itself equilateral.

The result is valid also when equilateral triangles $BX'C, CY'A, AZ'B$ are erected on the sides BC, CA, AB so as to be inwards and L', M', N' are the centres of the latter set of equilateral triangles $BX'C, CY'A, AZ'B$ respectively. Questions regarding the choice of equilateral triangles, some of which are outside and some inside and any distinguished properties of the resulting Torricelli configurations need to be examined with care.

The Torricelli configuration (consisting of the original triangle ABC, the external equilateral triangles BXC, CYA, AZB, the external Napoleon triangle LMN, the internal equilateral triangles $BX'C, CY'A, AZ'B$ and the internal Napoleon triangle $L'M'N'$) has several elegant properties.

1. The lines AX, BY, CZ that join the vertices A, B, C, and the opposite vertices X, Y, Z of the external equilateral triangles are concurrent at a point $(= p)$ which is internal to the triangle. It is called the inner isogonic point and is alternatively known as the inner FERMAT POINT of the triangle ABC. This point is such $B\hat{P}C = C\hat{P}A = APB = 2\pi/3$ and has the optimal property that

$$\min\ (QA + AB + QC)$$

$$QE \text{ interior of the triangle } ABC$$

occurs when Q is the inner Fermat point P. This minimum value equal $AX = BY = CZ$. More precisely we have $AX = BY = CZ = \in_1 PA + \in_2 PB + \in_3 PC$ where each $\in$ is either $+1$ on -1. A particular $\in$ equals -1 only when the angle of the triangle ABC at the concerned vertex exceeds $2\pi/3$.

2. The vertices ABC are the optical images of P in the sides MN, NL, LM respectively of the outer Napoleon triangle LMN.

3. The lines AL, BM, CN are concurrent.

4. Similar statements are valid in the case of the inward Napoleon triangle $L'M'N'$.

5. The outward Napoleon triangle LMN and the inward Napoleon triangle $L'M'N'$ have a common centre.

6. Area of the outward Napoleon triangle LMN - area of the inward Napoleon triangle LMN = area of the original triangle ABC.

7. The Fermat point P lies on the circum-circle of each of the equilateral triangles BXC, CYA, AZB and also on the circum-circle of the inner Napoleon triangle $L'M'N'$.

8. The inward Fermat point P' (at which the lines AX', BY', CZ' concur) lies on the circum-circle of each of the equilateral triangles $CX'B, AY'C, BZ'A$ and also on the circum-circle of the outer Napoleon triangle LMN.

9. AX', BY', CZ' make angles $= \pi/3$ with one another at the inward Fermat point P'.

10. $AX^1 = BY^1 = CZ^1 = \in_1 AP^1 + \in_2 BP^1 + \in_3 CP^1$ where each $\in = +1 or -1$.. It is -1 if the angle of the triangle ABC at the corresponding vertex equals $\pi/3$.

11. The point P is always external to the triangle ABC.

12. The vertices A, B, C are the mirror images of P in the sides M^1N^1, N^1L^1, L^1M^1 of the inward Napoleon triangle $L^1M^1N^1$.

13. It $\angle A \geq \pi/3, \angle B \geq \pi/3$ then P^1 (= the (outer) Fermat point) is the point Q external to the triangle ABC that maximizes the quantity $QC - QA - QB$.

14. When the angles at A and B are both less than $\pi/3$, the point Q which maximizes the quantity $QC - QA - QB$ coincides with the vertix of the smallest angle of the triangle ABC.

15. The triangle AYZ' and $AY'Z$ are both congruent to the original triangle ABC and the circum-centres of these triangles (AYZ^1) and (AY^1Z) lie on the circum-circle of the original triangle ABC.

16. Area $LMN+$ area $L^1M^1N^1$ = average of the areas of the three outward equilateral triangles BXC, CYA, AZB.

17. The Fermat points line, viz., PP^1 bisects the segment HG where H is the orthocentre and G is the Centroid of the original triangle ABC.

18. Let Q be the point such that the figure P^1HPQ is a parallelogram. Then Q lies on the circum-circle of the triangle ABC.

19. If lines are drawn through ABC so as to be perpendicular to PA, PB, PC respectively, the triangle formed by these lines is the smallest equilateral that can

be circumscribed about the original triangle. Further the area of the circumscribing triangle equals four times the area of the triangle ABC.

20. Let A_1A_2 be the points of trisection of the side BC of the triangle ABC let B_1B_2 and C_1C_2 be similarly defined on the sides CA and AB respectively. The figure $A_1A_2B_1B_2C_1C_2$ is a hexagon which is in general non-regular. If external and internal equilateral triangles are erected on the sides of the above hexagon, the summits of these two sets of equilateral triangles enclose regular hexagons.

HOW ARE THE THEOREMS OF TRIANGLE GEOMETRY DISCOVERED ?

It is a safe guess that results of the form mentioned above emerged after long and persistent hours of playing around the figure of the triangle. With the availability of accurate computer graphics, playing around the triangle is now an enjoyable pastime and it may lead to many results some of which may turn out to be new. Discoveries in triangle geometry through the use of computer graphics may be described as results arising from mathematical experimentation.

The discovery of Morley's trisector theorem in 1899 was on off-shoot of investigations on the metric geometry of the plane N - line.

Napoleon's theorem (which is attributed to the French General Napoleon Bonaparte) was (independently) discovered by an electrical engineer (Richard New-

mann) who used it in the theory of three phase alternating current circuits (1911).

The computer holds promise as an efficient device for mathematical experimentation in triangle geometry. Computer graphics and computer- assisted-geometric-industrial-design ($CAGID$) provide enough scope for playing around triangular configurations and in the wake of results centred round numerical parameters, some theorems may be inferred. Triangle geometry is a fertile practice ground for strategies of proof in the spirit of Euclid and it can also serve as the testing ground for strategies of proofs of theorems.

ADDITIONAL REMARKS

1. Morley's and Napoleon's theorems involve triangular configurations of which some are equilateral and this fact suggests that their role in mesh cutting of plane domains in the use of finite element methods of solution of physical problems needs to be explored in depth.

2. Torricelli configurations arise by the erection of equilateral triangles on the sides of an arbitrary triangle. What happens it triangles are erected on the sides BC, CA, AB of the triangle ABC and these erected triangles BXC, CYA, AZB are similar to one another ? What happens it these three triangles are all congruent with the original triangles and are reflections of the original triangle in the sides BC, CA, AB ?

3. Any optimal properties if BXC, CYA, AZB are all similar and each of these is isosceles with the vertical angles at X, Y, Z each equal to $\pi - 2\theta$?

4. Is it possible to secure the concurrence of the Euler lines of the three isosceles triangles BXC, CYA, AZB in (3) above by a suitable choice of θ ?

5. Are there any special results of significance or/of beauty arising from the configuration comprising the Morley triangles of the three isosceles triangles/ equilateral triangles BXC, CYA, AZB ?

These and many other such questions need to be examined using the technique of mathematical experimentation.

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